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MULTIPLY PERFECT NUMBERS OF FOUR DIFFERENT PRIMES *

BY R. D. CARMICHAEL

A MULTIPLY perfect number, according to the definition given by Lehmer † is one which is exactly divisible into the sum of its divisors including itself, and the quotient is called the multiplicity. It is easily shown that if the number is of the form

$$N = p_1^{a_1} p_2^{a_2} p_3^{a_3} p_4^{a_4}$$

with only four different prime factors, then the multiplicity m cannot be greater than 4.‡ The numbers of this form with multiplicity equal to 4 have already been discussed,§ and it was found that there is only one, namely $2^5 \cdot 3^3 \cdot 5 \cdot 7 = 30240$. There remain to be investigated therefore only those for which the multiplicity is 2 or 3, since the case $m = 1$ is excluded by the definition.

In the following pages it will be shown that the only even multiply perfect numbers of multiplicity 2 are of the known form $2^{a_1}(2^{a_1+1} - 1)$ where $2^{a_1+1} - 1$ is a prime, so that there are no even multiply perfect numbers of multiplicity 2 having four different primes. Furthermore it turns out that there is but one number of multiplicity 3, namely $2^9 \cdot 3 \cdot 11 \cdot 31$, and that there are no odd multiply perfect numbers of four different primes.

1. Even multiply perfect numbers of multiplicity 2. From the definition of a multiply perfect number it follows at once that

$$(1) \qquad m = \frac{p_1^{a_1} + p_1^{a_1-1} + \cdots + 1}{p_1^{a_1}} \cdots \frac{p_4^{a_4} + p_4^{a_4-1} + \cdots + 1}{p_4^{a_4}}$$
$$= \frac{p_1^{a_1+1} - 1}{p_1^{a_1}(p_1 - 1)} \cdots \frac{p_4^{a_4+1} - 1}{p_4^{a_4}(p_4 - 1)}.$$

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* Presented to the American Mathematical Society, October 27, 1906.

† ANNALS OF MATHEMATICS, ser. 2, vol. 2, p 103.

‡ See Lehmer, ANNALS OF MATHEMATICS, ser. 2, vol. 2, p. 1; or this can be seen from the inequality (2) by substituting 2, 3, 5, 7 for p_1, p_2, p_3, p_4 respectively.

§ Carmichael, *American Mathematical Monthly* , vol. 13, p. 88; 1906.

Consequently

$$m < \frac{p_1}{p_1 - 1} \cdots \frac{p_4}{p_4 - 1}. \tag{2}$$

We shall in the present section consider n factors instead of four. It is readily seen what (1) and (2) now become. In equation (1), putting $m = 2$, $p_1 = 2$ and reducing, we have

$$\frac{2^{\alpha_1+1}}{2^{\alpha_1+1} - 1} = \frac{p_2^{\alpha_2} + \cdots + p_2 + 1}{p_2^{\alpha_2}} \cdots \frac{p_n^{\alpha_n} + \cdots + p_n + 1}{p_n^{\alpha_n}}. \tag{3}$$

Hence

$$1 + \frac{1}{2^{\alpha_1+1} - 1} \underset{>}{=} \left(1 + \frac{1}{p_2}\right)\left(1 + \frac{1}{p_3}\right) \cdots \left(1 + \frac{1}{p_n}\right). \tag{4}$$

The only factors which $2^{\alpha_1+1} - 1$ can have are among the primes p_k. But if p_k is a factor of $2^{\alpha_1+1} - 1$,

$$1 + \frac{1}{p_k} > 1 + \frac{1}{2^{\alpha_1+1} - 1};$$

and hence (4) is not satisfied. Therefore $2^{\alpha_1+1} - 1$ must equal one of the p's. Hence one factor of the right member of (4) is equal to its left member. Therefore the right member cannot contain a second factor. Then $n = 2$ and $p_2 = 2^{\alpha_1+1} - 1$. Also, equation (3) can be satisfied only when $\alpha_2 = 1$. Hence,

All multiply perfect even numbers of multiplicity 2 *are of the form* $2^{\alpha_1}(2^{\alpha_1+1} - 1)$ *where* $2^{\alpha_1+1} - 1$ *is prime; and conversely all such numbers are multiply perfect numbers of multiplicity* 2.

2. Even multiply perfect numbers of the form $2^{\alpha_1} 3^{\alpha_2} p_3^{\alpha_3} p_4^{\alpha_4}$ **and multiplicity 3.** If $m = 3$ and $p_1 = 2$, it is easy to show from (2) that $p_2 = 3$ or 5. Hence we have as a first result that *if a number of the type* $2^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} p_4^{\alpha_4}$ *is multiply perfect the value of* p_2 *is either* 3 *or* 5. In this section we shall consider only the case when $p_2 = 3$. From equation (1) we have

$$3 = \frac{2^{\alpha_1+1} - 1}{2^{\alpha_1}} \cdot \frac{3^{\alpha_2+1} - 1}{2 \cdot 3^{\alpha_2}} \cdot \frac{p_3^{\alpha_3+1} - 1}{p_3^{\alpha_3}(p_3 - 1)} \cdot \frac{p_4^{\alpha_4+1} - 1}{p_4^{\alpha_4}(p_4 - 1)}. \tag{5}$$

For definiteness let us say that $p_4 > p_3$. We shall first determine some limitations which are to be placed on the exponents $a_3 + 1$ and $a_4 + 1$. The following lemma will be of frequent use :

Lemma. *If* x *is a positive integer* > 1, $x^t - 1$ *has a prime factor not dividing* $x^u - 1$ $(u < t)$, *except in the cases* $t = 2$, $x = 2^v - 1$, $v \geqq 2$; $t = 6$, $x = 2$.* *Such prime factors of* $x^t - 1$ *are of the form* $st + 1$, *and evidently if* t *is odd and* > 1 *they are of the form* $2st + 1$.† Such a prime $st + 1$ we shall call a characteristic factor.

Suppose that in equation (5) the exponent $a_4 + 1$ contains an odd prime factor > 1, say r, and let $a_4 + 1 = ry$, where y is to be determined. It may be shown by the lemma that in every case for which $y \neq 1$, $p_4^{ry} - 1$ contains more than one prime factor $\geqq 7$; for $p_4^r - 1$ and $p_4^{ry} - 1$ contain at least one characteristic factor each. But since in this case equation (5) can not be satisfied it follows that $y = 1$. That is, if $a_4 + 1$ contains an odd factor it must equal an odd prime. In the same way we may show that if $a_3 + 1$ contains an odd factor it must also be an odd prime.

It is evident at once from the theory of congruences that every prime divisor of $(p_4^{a_4+1} - 1) / (p_4 - 1)$, $a_4 + 1$ an odd prime, is a characteristic factor ; and therefore it readily follows from equation (5) that

$$p_4^{a_4+1} - 1 = (p_4 - 1)\, p_3^{a}, \tag{6}$$

where a is some integer $> a_4$, p_4 being $> p_3$. It is easy to show that when $a_3 + 1$ and $a_4 + 1$ each is an odd prime, neither of the expressions $(p_3^{a_3+1} - 1) / (p_3 - 1)$, $(p_4^{a_4+1} - 1) / (p_4 - 1)$ in (5) can contain either 2 or 3 ; for every prime factor of each is a characteristic factor $\geqq 7$. Then from (5) we have

$$1 = \left(1 - \frac{1}{2^{a_1+1}}\right)\left(1 - \frac{1}{3^{a_2+1}}\right) p_3^{\mu}\, p_4^{\nu}. \tag{7}$$

It is readily shown that this equation is not satisfied if either μ or ν is greater than zero or if both equal zero. Therefore it follows that one of the exponents $a_3 + 1$, $a_4 + 1$ can contain no prime factor except 2.

Let us take $a_3 + 1 = 2^{\mu}$ and let us still regard $a_4 + 1$ as equal to some odd prime integer. We should yet have as before equation (6) above ; and since

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* Dickson, "On the cyclotomic function," *American Mathematical Monthly*, vol. 12, p. 89, 1905.

† *Encyklopädie der Mathematischen Wissenschaften*, vol. 1, p. 577.

$\alpha > a_4$ we have $\alpha \geqq 3$. But $a_3 \geqq \alpha$ and therefore $a_3 + 1 \geqq 4$ and it follows that $\mu \geqq 2$. Moreover it is easy to show that if $a_4 + 1 \geqq 5$ then $\mu \geqq 3$. Now in every case for which $\mu \geqq 3$, the numerator $p_3^{a_3+1} - 1$ in equation (5) contains $p_3^8 - 1$. By the lemma, $p_3^4 - 1$, $p_3^8 - 1$ each contains a characteristic factor $\geqq 5$ and prime to p_3; but at most two primes $\geqq 5$ may be introduced and this case is therefore excluded, p_3 being $\geqq 5$. Then consider the case $\mu = 2$ and $a_4 + 1 = 3$. Now $a_3 = 3$, $a_4 = 2$, and $\alpha > a_4$ and $\leqq a_3$; and therefore $\alpha = 3$. Then from (6) we have

$$p_4^2 + p_4 + 1 = p_3^3.$$

Moreover every prime factor of $p_3^2 + p_3 + 1$ is a characteristic factor $\geqq 7$; and therefore p_4 is the only such factor. Hence

$$p_3^2 + p_3 + 1 = p_4 \text{ or } p_4^2,$$

neither case of which satisfies the next preceding equation. This case is therefore excluded. Therefore $a_3 + 1$ can not be a power of 2 when $a_4 + 1$ is not a power of 2. Combining this result with that of the preceding paragraph we are able to announce that $a_4 + 1 =$ some power of 2. In the same way as the condition $a_3 + 1 = 8$ or a multiple of 8 has just been excluded we may show that $a_4 + 1 \neq 8$ or a multiple of 8. *Therefore as a preliminary result we have that* $a_4 + 1$, *the exponent of* p_4, *is either* 2 *or* 4.*

The plan of proof in the following paragraphs of this section will be to show that $2^{a_1+1} - 1$ contains both the factors p_3 and p_4, and that therefore $a_2 = 1$, and then to examine that case. From equation (5) we have

$$(8) \qquad \frac{2^{a_1+1}}{2^{a_1+1}-1} \cdot \frac{3^{a_2+1}}{3^{a_2+1}-1} = \frac{p_3^{a_3} + \cdots + p_3 + 1}{p_3^{a_3}} \cdot \frac{p_4^{a_4} + \cdots + p_4 + 1}{p_4^{a_4}}.$$

Hence

$$(9) \qquad \left(1 + \frac{1}{2^{a_1+1}-1}\right)\left(1 + \frac{1}{3^{a_2+1}-1}\right) \geqq \left(1 + \frac{1}{p_3}\right)\left(1 + \frac{1}{p_4}\right).$$

The only possible prime factors of $2^{a_1+1} - 1$ are 3, p_3, p_4. If $2^{a_1+1} - 1$ contains the factor 3 it is readily seen, by aid of the congruence $2^{a_1+1} \equiv 1 \pmod 3$, that $a_1 + 1$ is a multiple of 2; say $= 2y$. If $y > 2$, characteristic factors of $2^y - 1$ and $2^{2y} - 1$ will necessarily be p_3 and p_4. If $y = 2$, the number

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* The reader will observe that practically the same reasoning applies with a similar conclusion to any case in which neither p_1 nor p_2 is greater than 5.

$2^{a_1+1}-1$ contains the prime 5, and therefore $p_3 = 5$; and then from inequality (9) it is easy to show that $p_4 > 3^{a_2+1} - 1$. Then $3^{a_2+1}-1$ contains no prime factors except 2 and 5, and therefore $a_2 = 1$ or 3. Then from (9) we should have

$$3 > \frac{3\cdot 5}{2^3}\cdot\frac{2^3\cdot 5}{3^3}\cdot\frac{2\cdot 3}{5},$$

which is not true; and hence $y \neq 2$. If $y = 1$, equation (5) may be reduced to

$$(10)\qquad 2 = \frac{3^{a_2+1}-1}{2\cdot 3^{a_2}}\cdot\frac{p_3^{a_3+1}-1}{p_3^{a_3}(p_3-1)}\cdot\frac{p_4^{a_4+1}-1}{p_4^{a_4}(p_4-1)},$$

a condition which can be satisfied only when $3^{a_2}p_3^{a_3}p_4^{a_4}$ is a multiply perfect odd number of three different primes. But it is known that no such numbers exist,* and hence *when 3 is a divisor of* $2^{a_1+1}-1$ *so also are* p_3 *and* p_4.

In the next case suppose that $2^{a_1+1}-1$ is not divisible by 3. Then a_1+1 is odd. We shall now show that $2^{a_1+1}-1$ is not a power of either p_3 or p_4. If

$$2^{a_1+1}-1 = p_i^a, \text{ we have } 2^{a_1+1} = p_i^a + 1 = \frac{p_i^{2a}-1}{p_i^a-1}.$$

Any characteristic factor of the numerator (when it has one) is by the lemma of the form $2as+1$, and hence > 2. The denominator does not contain this factor. Also $p_i^{2a}-1$ has always a characteristic factor except when $a = 1$ and p_i is of the form $2^v - 1$. Hence in every case $a = 1$, and therefore $2^{a_1+1}-1 = p_i$; that is, if $2^{a_1+1}-1$ is the power of a prime it is the first power. Then if $2^{a_1+1}-1$ is not divisible by 3 and contains only one of the primes p_3, p_4, say p_i, $2^{a_1+1}-1 = p_i$. Denote the other by p_k. Then by inequality (9), $p_k > 3^{a_2+1}-1$; hence $3^{a_2+1}-1$ is not divisible by p_k. By a second application of inequality (9) it is easy to show that if p_i divides $3^{a_2+1}-1$, then $p_i = 2^{a_1+1}-1 < p_k$; and therefore $p_i = p_3$ and $p_k = p_4$.

Proceeding to consider this case we should have $3^{a_2+1}-1 = xp_3^\mu$ where μ and x represent integers and $\mu < a_3$ and $x \geqq 2$. Then from (8) we have by a simple transformation

$$(11)\qquad \left(1+\frac{1}{p_3}\right)\left(1+\frac{1}{xp_3^\mu}\right) \geqq \left(1+\frac{1}{p_3}+\cdots+\frac{1}{p_3^{a_3}}\right),$$

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* Carmichael, *American Mathematical Monthly*, vol. 13, p. 35; 1906.

from which it follows that

$$(12) \qquad 1 + \frac{1}{p_3} + \frac{1}{xp_3^{\mu}} + \frac{1}{xp_3^{\mu+1}} \geqq 1 + \frac{1}{p_3} + \cdots + \frac{1}{p_3^{a_3}} .$$

But since both $2^{a_1+1} - 1$ and $3^{a_2+1} - 1$ contain p_3, $a_3 \geqq 2$. Moreover $\mu < a_3$ and also $x \geqq 2$. Hence inequality (12) is not satisfied except when $\mu = 1$. In this case we have $3^{a_2+1} - 1 = 2\,(2^{a_1+1} - 1)$; or $3^{a_2+1} = 2^{a_1+2} - 1$, an equation which cannot be satisfied since $a_1 \neq 0$. It follows therefore that $p_i = 2^{a_1+1} - 1$ is not a divisor of $3^{a_2+1} - 1$. But p_k does not divide $3^{a_2+1} - 1$ and therefore this number has no prime factor except 2 when $2^{a_1+1} - 1$ is a prime. Therefore $a_2 + 1 = 2$. Since $p_k > 3^{a_2+1} - 1$ it is equal to or greater than 11. Hence from (1) we may readily deduce, since $a_3 = 1$,

$$(13) \qquad 3 < \frac{2^{a_1+1} - 1}{2^{a_1}} \cdot \frac{4}{3} \cdot \frac{11}{10} \cdot \frac{2^{a_1+1} - 1}{2^{a_1+1} - 2} = \frac{44}{15}\, \frac{2^{2a_1+2} - 2^{a_1+2} + 1}{2^{2a_1+2} - 2^{a_1+2}} .$$

From this it follows readily that

$$2^{2a_1+2} - 2^{a_1+2} < 44,$$

and since $2^{a_1+1} - 1$ must be prime, it is easy to show that $2^{a_1+1} - 1 = 3$. This will lead again to an equation of the form of (10), and the case of this paragraph is therefore excluded. Hence *there exist no multiply perfect numbers of the type* $2^{a_1} 3^{a_2} p_3^{a_3} p_4^{a_4}$ *for which* $2^{a_1+1} - 1$ *is the power of a prime, so that when* $2^{a_1+1} - 1$ *is not divisible by* 3 *it contains both* p_3 *and* p_4. Combining this result with that of a preceding paragraph of this section we have that *for a multiply perfect number of the type* $2^{a_1} 3^{a_2} p_3^{a_3} p_4^{a_4}$ *and multiplicity* 3, $2^{a_1+1} - 1$ *must contain both* p_3 *and* p_4.

Now from inequality (9) it is easy to show by a method used in the preceding paragraphs that p_3 and p_4 each is greater than $3^{a_2+1} - 1$ and hence that $3^{a_2+1} - 1$ does not contain either p_3 or p_4. Then $3^{a_2+1} - 1$ must in every case be a power of 2, and hence by the lemma, $a_2 + 1 = 2$.

We proceed to investigate the values of $2^{a_1+1} - 1$ which contain two prime factors different from 3 and which might occur in equation (8). From (9) it follows readily that both p_3 and p_4 are $\geqq 11$, both being greater than $3^{a_2+1} - 1 = 8$, from (1) we have in the present case

$$(14) \qquad 3 < 2 \cdot \frac{4}{3} \cdot \frac{p_3}{p_3 - 1} \cdot \frac{p_4}{p_4 - 1},$$

from which it follows that $p_4 \leqq 43$. By inspection for smaller values of a_1 and with the help of the lemma for larger values, we find that there is only one value of $2^{a_1+1} - 1$, namely $2^{10} - 1 = 3 \cdot 11 \cdot 31$, which has two prime factors other than 3 and lying between 11 and 43, the limits 11 and 43 being included. For this value we have

$$(15) \quad 3 = \frac{3 \cdot 11 \cdot 31}{2^9} \cdot \frac{2^2}{3} \cdot \frac{11^{a_3} + \cdots + 11 + 1}{11^{a_3}} \cdot \frac{31^{a_4} + \cdots + 31 + 1}{31^{a_4}}.$$

We have above shown that $a_4 = 1$ or 3; and therefore by inspection of equation (15) it is easy to see that $a_4 = 1$. Likewise $a_3 = 1$ and the equation is satisfied; and we therefore have the result that *the number* $2^9 \cdot 3 \cdot 11 \cdot 31$ *is the only multiply perfect number of the type* $2^{a_1} 3^{a_2} p_3^{a_3} p_4^{a_4}$ *and of multiplicity* 3.

3. Non-existence of even multiply perfect numbers of the type $2^{a_1} 5^{a_2} p_3^{a_3} p_4^{a_4}$ and of multiplicity 3. In this case by the same reasoning as at the beginning of the preceding section,* it may be shown that $a_4 = 1$ or 3. With $m = 3$, $p_1 = 2$, $p_2 = 5$, it is readily seen from (2) that $p_3 = 7$ and $p_4 \leqq 31$. Then $2^{a_1+1} - 1$ can contain no prime > 31; also it has only one prime other than 3, 5, 7. Hence by a liberal use of the lemma it turns out that $a_1 + 1 = 2, 3, 4, 5, 6, 8$, or 12. The values 6 and 12 are excluded by their introducing too many 3's into the numerator of the right-hand member of (1). From (1) we may write

$$3 < \frac{2^{a_1+1} - 1}{2^{a_1}} \cdot \frac{5}{4} \cdot \frac{7}{6} \cdot \frac{p_4}{p_4 - 1}$$

By aid of this inequality the values $a_1 + 1 = 2, 3$ are readily excluded. There are left $a_1 + 1 = 4, 5, 8$, which we shall examine in turn.

For $a_1 + 1 = 4$, we have from (1)

$$(16) \quad 3 = \frac{3 \cdot 5}{2^3} \cdot \frac{5^{a_2} + \cdots + 5 + 1}{5^{a_2}} \cdot \frac{7^{a_3} + \cdots + 7 + 1}{7^{a_3}} \cdot \frac{p_4^{a_4} + \cdots + p_4 + 1}{p_4^{a_4}}.$$

Now no prime > 31 can be introduced. If $5^{a_2} + \cdots + 5 + 1$ contains 7 it contains 3, which by (16) is inadmissible. Hence $p_4^{a_4} + \cdots + 1$ must contain 7^{a_3}. If $p_4 = 11, 17, 19, 23, 29$, or 31, in each case we may readily show that an inadmissible prime will occur when $p_4^{a_4} + \cdots + p_4 + 1$ contains 7. Hence $p_4 = 13$, and since $a_4 = 1$ or 3, it is easy to show that $a_4 = 1$. Then $a_3 = 1$, and hence also $a_2 = 1$, and 3 occurs the second time in the numerator of (16); and this is inadmissible.

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* See footnote page 152.

For $a_1 + 1 = 5$, equation (1) becomes

$$3 = \frac{31}{2^4} \cdot \frac{5^{a_2} + \cdots + 5 + 1}{5^{a_2}} \cdot \frac{7^{a_3} + \cdots + 7 + 1}{7^{a_3}} \cdot \frac{31^{a_4} + \cdots + 31 + 1}{31^{a_4}}. \tag{17}$$

Since $a_4 = 1$ or 3 it is evident that the numerator of (17) contains too high a power of 2. In a similar manner we may show that when $a_1 + 1 = 8$ the resulting numerator contains too high a power of 3, p_4 in this case being equal to 17. This completes the discussion of numbers of the type $2^{a_1} 5^{a_2} p_3^{a_3} p_4^{a_4}$, no multiply perfect number being found. Combining the last result with those of section 2 we have the theorem that

The only even multiply perfect number of (only) four different primes and of multiplicity 3 *is the number* $2^9 \cdot 3 \cdot 11 \cdot 31$ *previously found.*

4. Non-existence of odd multiply perfect numbers of only four different primes. Let the number be $N = p_1^{a_1} p_2^{a_2} p_3^{a_3} p_4^{a_4}$ where $p_1 < p_2 < p_3 < p_4$ are different primes. Then by inequality (2) we see at once that for any odd multiply perfect number the multiplicity must be 2, while $p_1 = 3$ and $p_2 =$ either 5 or 7.

First let $p_2 = 7$. Then by (2), $p_3 = 11$ or 13.

If $p_3 = 13$, $p_4 = 17$ or 19, as may be readily shown by (2). For this case write (1) as follows:

$$2 = \frac{3^{a_1} + \cdots + 3 + 1}{3^{a_1}} \cdot \frac{7^{a_2} + \cdots + 7 + 1}{7^{a_2}} \cdot \frac{13^{a_3} + \cdots + 13 + 1}{13^{a_3}} \cdot \frac{p_4^{a_4} + \cdots + p_4 + 1}{p_4^{a_4}}.$$

By a liberal use of the lemma it may be shown from the last equation that $p_4 \neq 19$; and that if $p_4 = 17$, $a_4 = 1$. This introduces the only admissible 2 into the numerator above; moreover it is easy to show that if any numerator contains 17 the now inadmissible 2 is introduced, and hence this case yields no numbers of the type under consideration.

Let $p_3 = 11$. Then by (2) $p_4 \leqq 23$; and therefore $p_4 = 13, 17, 19$, or 23. By using the lemma we exclude the values 17, 19, 23; and if $p_4 = 13$

we may readily show that $a_4 = 1$. This introduces into the numerator of (1) the only admissible 2. The factor $11^{a_3} + \cdots + 11 + 1$ in (1) must contain 3, 7, or 13. But if it contains 3 or 13, the inadmissible 2 occurs; and if it contains 7, 19 is introduced making it impossible to satisfy (1). This case therefore yields no numbers of the type considered here. Hence we conclude that there are no multiply perfect numbers of the type $3^{a_1} 7^{a_2} p_3^{a_3} p_4^{a_4}$, $p_4 > p_3 > 7$.

We shall now take up the case when $p_2 = 5$. By the same reasoning as that which was used at the beginning of section 2, it follows that in this case also $a_4 = 1$ or 3. But if $a_4 = 3$, the numerator of the second member of (1) will contain too high a power of 2, and therefore $a_4 = 1$. Then from (1) we may write

$$(18)\quad 2 = \frac{3^{a_1} + \cdots + 3 + 1}{3^{a_1}} \cdot \frac{5^{a_2} + \cdots + 5 + 1}{5^{a_2}} \cdot \frac{p_3^{a_3} + \cdots + p_3 + 1}{p_3^{a_3}} \cdot \frac{p_4 + 1}{p_4}.$$

None of the quantities a_1, a_2, a_3 can be odd; for then the numerator of (18) would contain too many 2's. Now by the lemma the characteristic factors of $3^{a_1+1} - 1$, $5^{a_2+1} - 1$, $p_3^{a_3+1} - 1$ must each be > 5; for $a_1 + 1$, $a_2 + 1$, $a_3 + 1$ are all odd numbers $\gtreqless 3$. Then it is easy to see by inspection of equation (18) that

$$(19)\qquad p_3^{a_3} + \cdots + p_3 + 1 = p_4,$$

$$(20)\qquad (3^{a_1} + \cdots + 3 + 1)(5^{a_2} + \cdots + 5 + 1) = p_3^{\mu},$$

and then further that

$$(21)\qquad 2 \cdot 3^{a_1} \cdot 5^{a_2} p_3^{a_3 - \mu} = p_4 + 1.$$

From (20) it may be shown with little difficulty that

$$(22)\qquad 3^{a_1} 5^{a_2} \cdot \frac{3}{2} \cdot \frac{5}{4} > p_3^{\mu}; \quad \text{or} \quad 2 \cdot 3^{a_1} \cdot 5^{a_2} > \frac{16}{15} p_3^{\mu};$$

and then from (21) that

$$\frac{16}{15} p_3^{a_3} < p_4 + 1.$$

But from (19), $p_3^{a_3} \cdot \dfrac{p_3}{p_3 - 1} > p_4$, and hence $\dfrac{p_3}{p_3 - 1} > \dfrac{16}{15}$, from which it may be shown that $p_3 \gtreqless 13$. And now since p_3 is a characteristic factor of both $3^{a_1+1} - 1$ and $5^{a_2+1} - 1$, neither $a_1 + 1$ nor $a_2 + 1$ can be greater than 6; and since each is odd and > 1, the only possible values are 3, 5. Moreover these values are to be so selected that $3^{a+1}-1$ and $5^{a_2+1}-1$ shall have the common factor $p_3 = 7, 11,$ or 13, and neither shall have a factor $> p_3$. These conditions are not satisfied in any case. Combining this result with a previous one of this section we are able to announce that *there are no odd multiply perfect numbers of only four different primes.*

Collecting all the results of this paper and certain others referred to in the opening paragraphs, we have the theorem:

There exist but two multiply perfect numbers of (only) four different primes: these two are $2^5 \cdot 3^3 \cdot 5 \cdot 7$, *of multiplicity* 4; *and* $2^9 \cdot 3 \cdot 11 \cdot 31$, *of multiplicity* 3.

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